Stabilnost položaja ravnoteže na osnovu linearnosti

$x' = F(x)$

$x = 0$

$F(x) = \overset{0}{F(0)} + F'(0)x + R(x)$

$F(x) \approx F'(0)x$

$x' = \underbrace{F'(0)}_{A}x \Rightarrow x' = Ax$

1. Za koje vrijednosti parametara α i β je položaj ravnoteže $(0,0)$: $\begin{aligned} x_1' &= -\sin(x_1 + \alpha x_2) \\ x_2' &= \beta x_1 + \ln(1 - x_2) \end{aligned}$

asimptotski stabilan?

$\sin(x_1 + \alpha x_2) \approx x_1 + \alpha x_2$

$\ln(1 - x_2) \approx -x_2$

$\Rightarrow \quad x_1' = -x_1 - \alpha x_2$

$\quad\quad x_2' = \beta x_1 - x_2$

$A = \begin{pmatrix} -1 & -\alpha \\ \beta & -1 \end{pmatrix}$

$\det(A - \lambda E) = \begin{vmatrix} -1-\lambda & -\alpha \\ \beta & -1-\lambda \end{vmatrix} = (1+\lambda)^2 + \alpha\beta = 0$

$= 1 + 2\lambda + \lambda^2 + \alpha\beta = 0$

$\lambda_{1,2} = -1 \pm \sqrt{-\alpha\beta}$

1° $-\alpha\beta \leq 0$

$\lambda_{1,2} = -1 \pm i\sqrt{\alpha\beta} < 0 \Rightarrow$ asimptotski stabilan

2° $-\alpha\beta > 0$

$\lambda_{1,2} = -1 \pm \sqrt{-\alpha\beta}$

$\lambda_1 = -1 - \sqrt{-\alpha\beta} < 0$

$\lambda_2 = -1 + \sqrt{-\alpha\beta} < 0 \quad , \quad \sqrt{-\alpha\beta} < 1$

$0 < -\alpha\beta < 1$

$\alpha\beta > -1$ - uključujemo -1 jer je nula višestr. 1

2. $x_1' = \alpha x_1 - \cos x_2 + e^{\beta x_3}$

$x_2' = \beta \sin x_1 + \ln(1 + \alpha x_2) - x_1 x_3^2$

$x_3' = x_1^2 \cos x_3 + \beta x_2 + \sin \alpha x_3$

$\cos x_2 = 1 - \sin 0 \cdot x_2 + O(x_2)$

$e^{\beta x_3} = 1 + \beta x_3 + O(x_3)$

$\beta \sin x_1 \approx \beta x_1$

$\ln(1 + \alpha x_2) \approx \alpha x_2$

$\dfrac{x_1 x_3^2}{\|x\|} \xrightarrow[\|x\| \to 0]{} 0$

$\dfrac{x_1 x_3^2}{\max\{x_1, x_3\}} \longrightarrow 0 \Rightarrow x_1 x_3^2 = O(x)$

u odnosu na bilo koju normu u $\mathbb{R}^2$

$x_1^2 \cos x_3 \approx x_1^2 = O(x_1)$

$\sin \alpha x_3 \approx \alpha x_3$

$x_1' = \alpha x_1 - 1 + 1 + \beta x_3 \Rightarrow x_1' = \alpha x_1 + \beta x_3$

$x_2' = \beta x_1 + \alpha x_2$

$x_3' = \beta x_2 + \alpha x_3$

$$A = \begin{pmatrix} \alpha & 0 & \beta \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} \alpha-\lambda & 0 & \beta \\ \beta & \alpha-\lambda & 0 \\ 0 & \beta & \alpha-\lambda \end{vmatrix} = (\alpha-\lambda)^3 + \beta^3 = 0$$

$$-\lambda^3+3\alpha\lambda^2-3\alpha^2\lambda+\alpha^3+\beta^3=0$$

$$\lambda^3-3\alpha\lambda^2+3\alpha^2\lambda-\alpha^3-\beta^3=0$$

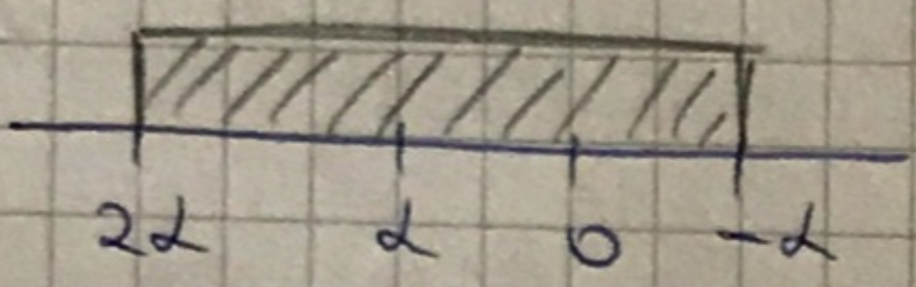
$$\begin{pmatrix} -3\alpha & 1 & 0 \\ -\alpha^3-\beta^3 & 3\alpha^2 & -3\alpha \\ 0 & 0 & -\alpha^3-\beta^3 \end{pmatrix}$$

$$\Delta_1 = -3\alpha > 0 \Rightarrow \alpha < 0$$

$$\Delta_2 = \begin{vmatrix} -3\alpha & 1 \\ -\alpha^3-\beta^3 & 3\alpha^2 \end{vmatrix} = -9\alpha^3+\alpha^3+\beta^3 =$$

$$= -8\alpha^3+\beta^3 > 0 \Rightarrow \beta^3 > 8\alpha^3$$

$$\beta > 2\alpha$$



$$\Delta_3 = \underbrace{(-\alpha^3-\beta^3)}_{>0}\overset{>0}{\Delta_2}$$

$$-\alpha^3-\beta^3>0 \Rightarrow \beta < -\alpha$$

$$\alpha<0, \quad 2\alpha<\beta<-\alpha$$

3. $x' = -\sin y$

$y' = 2x+\sqrt{1-3x-\sin y}$

$$\left.\begin{aligned} &-\sin y = 0 \\ &2x+\sqrt{1-3x-\sin y}=0 \end{aligned}\right\} \quad y = k\pi, \ k\in\mathbb{Z}$$

$$2x+\sqrt{1-3x}=0$$

$$\sqrt{1-3x} = -2x \quad /^2$$

$$1-3x = 4x^2$$

$$4x^2+3x-1=0$$

$$x_{1,2} = \frac{-3\pm\sqrt{9+16}}{8}$$

$$x_1 = -1, \quad x_2 = \frac{1}{4}$$

$$\frac{1}{2}+\sqrt{1-\frac{3}{4}} = \frac{1}{2}+\frac{1}{2} \neq 0$$

$(-1, k\pi)$, $k\in\mathbb{Z}$

$$x = -1+\varepsilon_1 \Rightarrow x' = \varepsilon_1'$$

$y = k\pi + \varepsilon_2 \Rightarrow y' = \varepsilon_2'$

$\varepsilon_1' = -\sin(k\pi + \varepsilon_2)$

$\varepsilon_2' = 2(\varepsilon_1 - 1) + \sqrt{1 - 3(\varepsilon_1 - 1) - \sin(k\pi + \varepsilon_2)}$

$(0,0)$ - položaj ravnoteže

$-\sin(k\pi + \varepsilon_2) = -(\underbrace{\sin k\pi}_{=0} \cdot \cos\varepsilon_2 + \underbrace{\cos k\pi}_{=(-1)^k} \cdot \sin\varepsilon_2) =$

$= -(-1)^k \sin\varepsilon_2 \approx (-1)^{k+1}\varepsilon_2$

$\sqrt{1 - 3\varepsilon_1 + 3 - \sin(k\pi + \varepsilon_2)} \approx (4 - 3\varepsilon_1 + (-1)^{k+1}\varepsilon_2)^{\frac{1}{2}}$

$(1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x$

$= 2\left(1 - \frac{3}{4}\varepsilon_1 + \frac{1}{4}(-1)^{k+1}\varepsilon_2\right)^{\frac{1}{2}} \approx 2\left(1 - \frac{3}{4\cdot 2}\varepsilon_1 + \frac{1}{4\cdot 2}(-1)^{k+1}\varepsilon_2\right) =$

$= 2 - \frac{3}{4}\varepsilon_1 + \frac{1}{4}(-1)^{k+1}\varepsilon_2$

$$\begin{cases} \varepsilon_1' = (-1)^{k+1}\varepsilon_2 \\ \varepsilon_2' = 2\varepsilon_1 - 2 + 2 - \frac{3}{4}\varepsilon_1 + \frac{1}{4}(-1)^{k+1}\varepsilon_2 \end{cases}$$

$\varepsilon_1' = (-1)^{k+1}\varepsilon_2$

$\varepsilon_2' = \frac{5}{4}\varepsilon_1 + \frac{1}{4}(-1)^{k+1}\varepsilon_2 \Rightarrow A = \begin{pmatrix} 0 & (-1)^{k+1} \\ \frac{5}{4} & \frac{1}{4}(-1)^{k+1} \end{pmatrix}$

$1^\circ \quad k = 2n$

$$A = \begin{pmatrix} 0 & -1 \\ \frac{5}{4} & -\frac{1}{4} \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & -1 \\ \frac{5}{4} & -\frac{1}{4} - \lambda \end{vmatrix} = \lambda^2 + \frac{\lambda}{4} + \frac{5}{4} = 0$$

$$\lambda_{1,2} = \frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 5}}{2}$$

$$Re(\lambda_{1,2}) = -\frac{1}{8} < 0$$

$\Rightarrow (-1, 2n\pi)$ asimptotski stabilan

2° $k = 2n+1$

$$A = \begin{pmatrix} 0 & 1 \\ \frac{5}{4} & \frac{1}{4} \end{pmatrix}$$

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & 1 \\ \frac{5}{4} & \frac{1}{4} - \lambda \end{vmatrix} = \lambda^2 - \frac{4}{5}\lambda - \frac{5}{4} = 0$$

$$\lambda_{1,2} = \frac{\frac{1}{4} \pm \sqrt{\frac{1}{16} + 5}}{2}$$

$$\lambda_1 = \frac{\frac{1}{4} + \sqrt{\frac{1}{16} + 5}}{2} > 0$$

$\Rightarrow$ $(-1, (2n+1)\pi)$ - nestabilan, $n \in \mathbb{Z}$

4. $x_1' = e^{x_1} - e^{3x_3}$

$x_2' = 4x_3 - 3\sin(x_1 + x_2)$

$x_3' = \ln(1 - 3x_1 + x_3)$

5. $x_1' = \ln(e + ax_1) - e^{x_2}$

$x_2' = 6x_1 + \operatorname{tg} x_2$

$(x_1, x_2) = (0, 0)$